

How many vertex locations can be arbitrarily chosen when drawing planar graphs?

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Abstract

It is proven that every set S of distinct points in the plane with cardinality $\left\lceil \frac{\sqrt{\log_2 n - 1}}{4} \right\rceil$ can be a subset of the vertices of a crossing-free straight-line drawing of any planar graph with n vertices. It is also proven that if S is restricted to be a one-sided convex point set, its cardinality increases to $\lceil \sqrt[3]{n} \rceil$. The proofs are constructive and give rise to $O(n)$ -time drawing algorithms. As a part of our proofs, we show that every maximal planar graph contains a large induced biconnected outerplanar graphs and a large induced outerpath (an outerplanar graph whose weak dual is a path).

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1 Introduction and Overview

Computing drawings of planar graphs is an extensively studied subject. In its most classical setting, the problem is to design an algorithm that receives as input a planar graph G and it produces as output a drawing of G such that no two edges of the drawing cross. Aesthetic requirements, such as minimizing the number of bends along the edges, using a small area, or representing the faces as convex or star-shaped polygons can be specified as additional optimization goals. In addition to these geometric optimization goals, a graph drawing algorithm can receive as input a set of semantic constraints which define placement and/or routing requirements for some of the vertices and/or some of the edges. For example, a subset of the vertices must be collinear, some of the edges must be horizontal, the size or shape of some edges or vertices is defined in advance (see, e.g., [10, 17, 25]).

A well studied graph drawing problem with semantic constraints is the point set embeddability problem. Given a planar graph G with n vertices and a set S of n distinct points in the plane, the problem is to compute a crossing-free drawing of G such that its vertices are mapped to the points of S . Kaufmann and Wiese [18] proved that any planar graph has a point-set embedding with at most two bends per edge on any given point set. The drawing may however use exponential area, and a polynomial area construction is showed in [11]. If no bends along the edges are allowed, Cabello [8] proved that deciding whether a planar graph admits a straight-line point-set embedding is an NP-complete problem. Nishat et al. [22] and Moosa and Rahman [21] presented polynomial-time algorithms to test straight-line point-set embeddability of plane 3-trees. At last year SoCG, Biedl and Vatschelle [4] showed that the problem can be solved in polynomial time for planar graphs with a fixed combinatorial embedding that have constant treewidth and constant face-degree. They also show that if one of the conditions is dropped (i.e., either the treewidth is unbounded or some faces have large degrees), the problem becomes NP-hard. Gritzman, Mohar, Pach and Pollack [15] and independently Castañeda and Urrutia [9] proved that the outerplanar graphs with n vertices are the largest family of graphs admitting a straight-line point-set embedding on any set of n points in general position.

Motivated by the above complexity results and restrictions on the classes of planar graphs that admit a geometric point-set embedding, we study the problem of drawing a planar graph with specified vertex locations from a different angle. Namely, we consider a scenario in which the given locations are fewer than the vertices of the graph. In our setting the algorithm receives as input any planar graph G with n vertices and any “sufficiently small” set S of points. The output is a crossing-free straight-line drawing such that $|S|$ vertices of G are mapped to the points of S , while the points representing the remaining $n - |S|$ vertices of G are chosen by the algorithm. We call the output a *geometric point-subset embedding of G on S* . We stress that S can be any set of points, i.e. the drawing algorithm must work for all point sets of “sufficiently small” cardinality. We are interested in making the “sufficiently small” cardinality as large as possible. For example, it is immediate to see that all planar graphs with at least four vertices have a geometric point-subset embedding of G on all point sets of cardinality three. Less intuitively, it has been proven in [1] that all planar graphs with at least six vertices have a geometric point-subset embedding of G on all point sets of cardinality four. The following significantly improves this cardinality.

Theorem 1 *Let G be an n -vertex plane graph with vertices and let S be a set of distinct points in the plane such that $|S| \leq \left\lceil \frac{\sqrt{\log_2 n - 1}}{4} \right\rceil$. G has a geometric point-subset embedding on S , which can be computed $O(n)$ time.*

We also prove that if the points are in one-sided convex position (i.e. they form a convex point

set such that the leftmost and the rightmost points are adjacent in the convex hull), the bound on the cardinality in Theorem 1 can be further increased.

Theorem 2 *Let G be an n -vertex plane graph and let S be a one-sided convex point set in the plane such that $|S| \leq \lceil \sqrt[3]{n} \rceil$. G has a geometric point-subset embedding on S , which can be computed $O(n)$ time.*

We recall that in [1] it is proven that there exists a one-sided convex point set S of size $\lceil \sqrt{n} \rceil$ such that any planar graphs admits a geometric point-subset embedding on S . Theorem 2 extends this result to all one-sided convex point sets, even if with a slightly smaller cardinality.

Overview of the proof technique. The proofs of Theorems 1 and 2 follow a similar approach, although they use different combinatorial results. The general idea is described in the following steps. Let G be an n -vertex plane graph (i.e. a planar graph with a given planar embedding) and let S be a set of k points with $k < n$.

Step 1: Graph G is augmented to a maximal plane graph G' with n vertices.

Step 2: An induced subgraph $H \subset G'$ of size at least k is computed.

Step 3: A point subset embedding Γ_H of H on S is computed.

Step 4: A planar embedding preserving straight-line drawing of G' is constructed by adding vertices and edges to Γ_H .

Step 5: The edges of $G' - G$ are removed.

Concerning Step 2, the induced subgraph H is outerplanar in the proofs of both theorems. In the proof of Theorem 2, H is a biconnected outerplanar graph having at least $\lceil \sqrt[3]{n} \rceil$ vertices; in the proof of Theorem 1, H is an outerpath (i.e. a biconnected outerplanar graph whose weak dual is a path) having at least $\left\lceil \frac{\sqrt{\log_2 n - 1}}{4} \right\rceil$ vertices. The existence of these subgraphs will be proven in Theorems 5 and 6, where it is also showed how to compute them in $O(n)$ time. The arguments combine a variety of concepts, including Schnyder’s realizers [24], rectangular representations [26], stabbing techniques [27], 4-block trees [28], and Mirsky’s theorem [20]. As for Steps 3 and 4, they rely on different drawing techniques concerning point-set embeddings and straight-line drawings of plane graphs inside star-shaped polygons [16].

We conclude this introduction by remarking that Theorems 5 and 6 may be of independent interest. Namely, proving the existence of large induced subgraphs has a long tradition in combinatorial graph theory. Erdős, Saks, and Sós [12] asked what is the maximum size of an induced tree in a graph and studied the relation between such size and other natural graph parameters. For example, they give upper and lower bounds on large induced trees in K_r -free graphs; these bounds have been improved by Matoušek and Šámal [19] and by Fox, Loh, and Sudakov [13]. Restricting to paths, Aroha and Valencia [2] showed that every 3-connected planar graph with a large number of vertices has a long induced path. More specifically, they proved that if a 3-connected graph G has a vertex of degree d , then G contains an induced path of length at least $\sqrt{\log_3 d}$. Generalizing this result to non-planar graphs, Böhme et al. [5] proved that for every positive integers k , r and s there exists an integer $n = n(k, r, s)$ such that every k -connected graph of order at least n contains either an induced path of length s or a subdivision of the complete bipartite graph $K_{k,r}$. However, their construction, do not seem to yield neither an explicit function for the length of the path nor

an algorithm to compute it. An implication of our results is that every n vertex maximal planar graph has an induced path of length at least $\left\lceil \frac{\sqrt{\log_2 n - 1}}{4} \right\rceil$, which can be computed in $O(n)$ time.

The rest of the paper is organized as follows. In Section 2 we give preliminary definitions. The techniques behind the proofs of Theorem 1 and 2 are described in Sections 3 and 4, respectively. Finally, open problems can be found in Section 5.

2 Preliminaries

Let G be a graph. A *drawing* Γ of G is a mapping of each vertex v of G to a point p_v in the plane, and of each edge $e = (u, v)$ to a Jordan arc connecting p_u and p_v not passing through any other vertex. Γ is *straight-line* if its edges are drawn as straight-line segments. Γ is *planar* if no two edges intersect (except possibly at common endpoints). A planar drawing Γ partitions the plane into topologically connected regions called *faces*; the unbounded region is the *external* face and the other regions are the *internal* faces. The boundary of a face is its delimiting circuit. All the face boundaries of a biconnected graph are simple circuits. A planar drawing determines a circular ordering of the edges around each vertex. The cyclic ordering of the edges around each vertex of Γ together with a choice of the external face is a *planar embedding* of G . A *plane graph* is a graph with a fixed planar embedding. The boundary of the external face of a plane graph G will be also called the *external boundary* of G . The *dual graph* G^* of a given plane graph G is a plane graph that has a vertex corresponding to each face of G , and an edge joining two vertices corresponding to neighboring faces of G . The *weak dual* of G is the graph obtained from G^* after removing the vertex that corresponds to the external face of G . A 3-cycle of a graph G is also called a *triangle* of G . A planar (plane) graph is *maximal* if each face of the graph is a triangle, thus no edge can be added to it without violating planarity. A subgraph H of a graph G is said to be *induced* if, for any pair of vertices u and v of H , (u, v) is an edge of H if and only if (u, v) is an edge of G .

Let G be a graph. A *separating k -ple* of G is a set of k vertices whose removal disconnects G . A graph G is *k -connected* if it does not contain a separating $(k - 1)$ -ple. Every maximal planar graph is 3-connected and every separating 3-ple of G (if any) is a triangle of G . Let t_1 and t_2 be two separating triangles of a maximal plane graph G ; we say that t_2 is *nested* inside t_1 and write $t_1 \succ t_2$ if at least one vertex of t_2 is inside t_1 in the planar embedding of G .

A graph is *outerplanar* if it admits a planar embedding where all vertices are on the external boundary. An *outerpath* is an outerplanar graph whose weak dual is a path. Let H be an outerpath and let f_1 and f_2 be the two faces corresponding to the endpoints of H^* . The edges of f_1 and f_2 that belong to the external boundary of H are called the *extremal edges* of H . Let e_1 be an extremal edge on the boundary of f_1 , let e_2 be an extremal edge on the boundary of f_2 . Any path in the external boundary of H that connects an endvertex of e_1 with an endvertex of e_2 is called a *side path* of H . It is easy to see that two extremal edges e_1 and e_2 define two side paths π_1 and π_2 and that every edge of H that is not on the external boundary has an endvertex in π_1 and an endvertex in π_2 . In other words both π_1 and π_2 are induced paths. Let e be an extremal edge of H ; any side path starting at an endvertex of e is also called a *side path of H with respect to e* . Let G be a maximal plane graph that contains an induced outerpath subgraph H with an extremal edge e on the external boundary of G and such that there is a side path of H with respect to e that has k vertices. We say that G *contains an external outerpath H with a side path of size k* .

Let S be a set of k points on the plane, $k > 0$ and let G be a planar graph with n vertices, with $k \leq n$. A *geometric point-subset embedding* of G on S is a planar straight-line drawing Γ of G such that each point of S represents a vertex of G . If G is a plane graph and the planar embedding of G

defined by Γ is the same as the one of G , we say that Γ *preserves* the planar embedding of G . All the algorithms of this paper compute drawings that preserve the planar embedding of the input graph. So when we write “geometric point-subset embedding” we mean “geometric point-subset embedding that preserves the planar embedding”.

3 General Point Sets

In this section we prove Theorem 1. The main ingredients of the proof are: (i) A drawing algorithm that, given a set S of k points and a maximal plane graph G that contains an external outerpath with a side path of size k , computes a geometric point-subset embedding of G on S (Section 3.1). (ii) An algorithm that, given a maximal plane graph G , computes an external outerpath of G with a side path of size at least $\left\lceil \frac{\sqrt{\log_2 n - 1}}{4} \right\rceil$ (Section 3.2).

3.1 Drawing Tools

Let P be a polygon on the plane, a *kernel* $K(P)$ of polygon P is the set of all points inside P from which all vertices of P are visible. We say that P is *star-shaped* if $K(P) \neq \emptyset$.

Theorem 3 (Hong, Nagamochi [16]) *Let G be an n -vertex 3-connected planar graph and let C be its external boundary. Every planar straight-line drawing of C as a star-shaped polygon can be extended to a planar straight-line drawing of G , where each internal face is represented by a convex polygon. Such a drawing can be computed in $O(n)$ time.*

Lemma 1 *Let S be a set k distinct points in the plane and let G be an n -vertex maximal plane graph that contains an external outerpath with a side path of size at least k . G admits a geometric point-subset embedding on S , which can be computed in $O(k \log k + n)$ time.*

Proof. Assume that the points of S have distinct x -coordinates. If not, we can rotate the plane so to achieve this condition. Let p_1, p_2, \dots, p_k be the points of S according to their left-to-right order and let (x_i, y_i) be the coordinates of p_i ($i = 1, 2, \dots, k$). Without loss of generality assume $x_1 = 0, y_1 = 0$. If this is not the case we can translate the plane to achieve this condition. Let P be the side path of the external outerpath H of G that has size at least k . If P has more than k vertices we can augment S with additional points. So without loss of generality assume that P has exactly k vertices. Let u_1, u_2, \dots, u_k be the vertices of P in the order they appear along P with u_1 being the vertex of P on the external boundary of G . Map u_i to p_i and draw each edge (u_i, u_{i+1}) as the segment $\overline{p_i p_{i+1}}$ (see Figure 1(a) for an illustration of the drawing technique). All vertices of H that are not in P form a second side path P' of H . Since P' is also an induced path, all the edges of H that are not in P nor in P' connect vertices of P to vertices of P' . Let v_1, v_2, \dots, v_h be the vertices of P' in the order they appear along P' with v_h being the vertex of P' on the external boundary of G . We want to place these vertices below the points of S in such a way that each point of S can be connected to the vertices of P' without crossing the segments $\overline{p_i p_{i+1}}$. Let $\delta_x = \min_i \{|x_{i+1} - x_i|\}$, let $\Delta_y = \max_{i,j} \{|y_i - y_j|\}$, and let $\sigma = (\Delta_y + 1)/\delta_x$. Let σ_i be the slope of segment $\overline{p_i p_{i+1}}$ ($i = 1, 2, \dots, k-1$). We have $-\sigma < \sigma_i < \sigma$ for every $i = 1, 2, \dots, k-1$. We place vertex v_i at point $q_i = (x_k + 1, -\sigma(x_k + i))$. The slope of each segment $\overline{p_i q_j}$ is at most $-\sigma$ and therefore each $\overline{p_i q_j}$ does not cross any segment $\overline{p_i p_{i+1}}$. Also, segments $\overline{p_i q_j}$ do not cross each other. Namely, let $\overline{p_{i_1} q_{j_1}}$ and $\overline{p_{i_2} q_{j_2}}$ be two such segments. Since the left-to-right order of the vertices of P in the drawing is the order in which they appear along P going from the external boundary to the

interior, and the bottom-to-top order of the vertices of P' in the drawing is the order in which they appear along P' going from the external boundary to the interior, then $i_1 \leq i_2$ and $j_2 \leq j_1$. This means that the slope of $\overline{p_{i_1}q_{j_1}}$ is less than the slope of $\overline{p_{i_2}q_{j_2}}$; since q_{j_1} is below q_{j_2} , the two segments do not cross each other. The drawing computed so far is a geometric point-subset embedding of H on S . Also, the drawing preserves the embedding of H . Let w be the unique vertex of the external boundary that does not belong to H . In order to complete the drawing we place the vertex w at point $(x_k + 2, \sigma(x_k + 2))$ and draw all the edges connecting it to all its adjacent vertices that have already been drawn. Also in this case, the added edges do not cross the existing ones. So the obtained drawing Γ' is a geometric point-subset embedding of the graph H' induced by the vertices in H plus vertex w . and also in this case the planar embedding is preserved. It is also easy to see that all the faces of Γ' are star-shaped polygons. Notice that, in the planar embedding of G each vertex not yet drawn is inside one of the cycles represented by the faces of H' . Let C be one of these cycles and let G' be the graph induced by the vertices of C plus all the vertices that are inside C in the planar embedding of G . All the internal faces of G' are triangles, while its external boundary is C (which therefore can have more than three vertices). Notice that there is no edge connecting two non-consecutive vertices of the external boundary of G' ; namely if such an edge existed it would have been drawn in Γ' and therefore C would not have been a cycle of H' . An internally triangulated graph is 3-connected if and only if there is no edge connecting two non-consecutive vertices of its external boundary (see, for example, [3]). Hence, G' is 3-connected and its external boundary is drawn as a star-shaped polygon. By Theorem 3 the drawing of C can be extended to a planar drawing of G' . Since this is true for all cycles that contain some not yet drawn vertices, the drawing Γ' can be completed to a geometric point-subset embedding Γ of G on S . Since a 3-connected planar graph has a unique embedding once the external face is fixed, the completion of Γ' to Γ preserves the embedding of G .

Concerning the time complexity, it is easy to see that the drawing can be computed in $O(n)$ time once the input points are sorted from left to right. Such a sorting preprocessing requires time $O(k \log k)$. \square

Lemma 2 *Let G be an n -vertex maximal plane graph. Let u_1, u_2 , and u_3 be the three external vertices of G and let u'_1, u'_2 , and u'_3 be three vertices of an internal face f such that there exists three induced paths $u_1 - u'_1$, $u_2 - u'_2$, and $u_3 - u'_3$ in G (possibly of zero length). Let τ be a straight-line drawing of f on any set S of three points. G admits a geometric point-subset embedding on S that has τ as a subdrawing. Such a point-subset embedding can be computed in $O(n)$ time.*

Proof. Let p_1, p_2 , and p_3 be the three points representing u_1, u_2 , and u_3 , respectively. Let ρ_i be the bisector of the angle at p_i of triangle τ ($0 \leq i \leq 2$), and let ρ_i^+ be the half-line contained in ρ_i that is completely outside τ (except for point p_i). Let $u_i = v_{i,0}, v_{i,1}, \dots, v_{i,k_i} = u'_i$ be the vertices of the path $u_i - u'_i$ ($0 \leq i \leq 2$). Place vertex $v_{i,j}$ along ρ_i^+ at distance j from p_i (see Figure 1(b) for an illustration of the drawing technique). Since each of the paths $u_i - u'_i$ is induced, there is no edge connecting non consecutive vertices along the path, and therefore each path can be drawn as a subdivision of a segment. We also add to the drawing all segments representing edges connecting the vertices of the three induced paths. Since the angle between ρ_i and ρ_i^+ is less than π , and since the vertices of each path $u_i - u'_i$ are placed along ρ_i^+ in the same order they have along the path $u_i - u'_i$, the resulting drawing is planar. Moreover, each face of the drawing is drawn as a star-shaped polygon. In the planar embedding of G each vertex not yet drawn is inside one of the cycles represented by the faces of the computed drawing. Thus, the drawing can be completed by using Theorem 3 as described in the proof of Lemma 1. It is immediate to see that the computed drawing preserves the embedding and that the algorithm time complexity is $O(n)$. \square

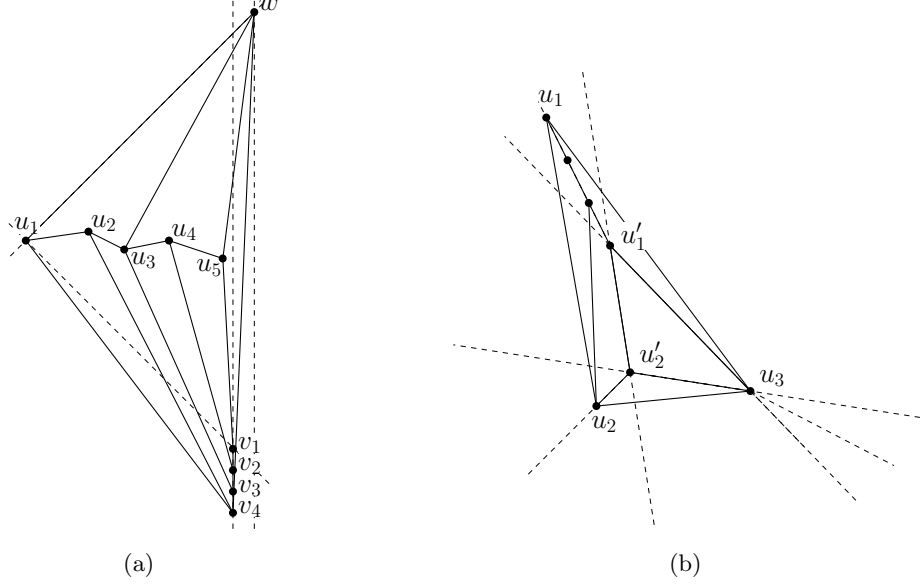


Figure 1: (a) Illustration of the drawing technique of Lemma 1. (b) Illustration of the drawing technique of Lemma 2.

3.2 Large Induced Outerpath

We now show that each n -vertex maximal plane graph contains an external outerpath with a side path of size $\left\lceil \frac{\sqrt{\log_2 n - 1}}{4} \right\rceil$. Our proof is based on the fact that an n -vertex maximal plane graph contains either many nested separating triangles or a large 4-connected subgraph. Thus, we first concentrate on graphs with many separating triangles and then on 4-connected graphs.

3.2.1 Large Induced Outerpath in Graphs with Many Nested Triangles

In order to prove that a graph with many separating triangles contains a large induced outerpath, we need to introduce the Schnyder realizer [24], which will also be used in Section 4.

Schnyder realizers. Let G be a maximal plane graph with $n \geq 3$ vertices. A *Schnyder realizer* [24] of G is an orientation plus a coloring of the internal edges of G such that:

1. Each internal edge receives one of three colors 0, 1, and 2;
2. Each internal vertex has exactly one outgoing edge of each color;
3. The counterclockwise order of the outgoing edges around each internal vertex is 0, 1, 2;
4. For each internal vertex, the incoming edges of a color appear counterclockwise between the two outgoing edges of the other two colors.

See Figure 2(a) for an example of a Schnyder realizer of a maximal plane graph equipped with. Note that conditions 3 and 4 of the definition of a Schnyder realizer imply that the counterclockwise circular ordering of the edges around each internal vertex is the one shown in Figure 2(b), i.e., outgoing colored 0, incoming colored 2, outgoing colored 1, incoming colored 0, outgoing colored 2, incoming colored 1, where each of the incoming set can be empty. It is known that every maximal

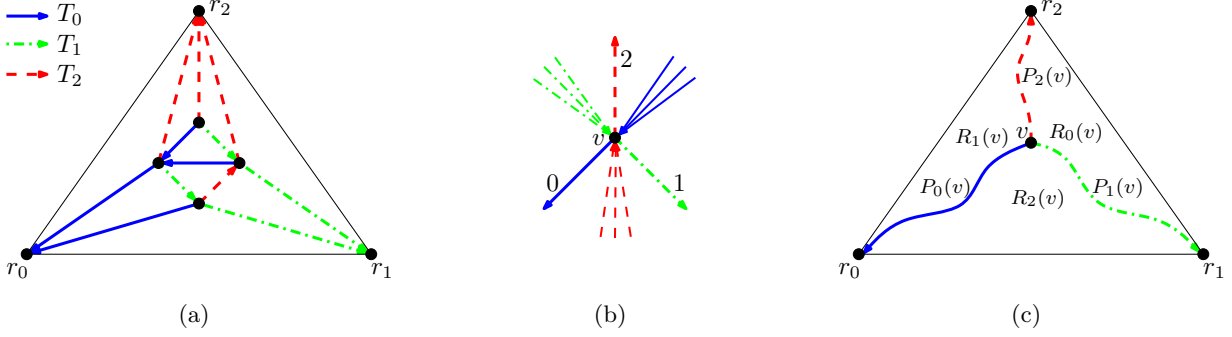


Figure 2: (a) A Schnyder realization $\mathcal{T} = \{T_1, T_2, T_3\}$ of a maximal plane graph. (b) The circular ordering of the edges around an internal vertex v . (c) The three paths $P_1(v)$, $P_2(v)$, and $P_3(v)$ and the three regions $R_1(v)$, $R_2(v)$, and $R_3(v)$

planar embedded graph has a Schnyder realization. Let T_0 , T_1 , and T_2 be the subgraph induced by the edges of color 0, 1, and 2, respectively. It is possible to prove each of T_0 , T_1 , and T_2 is a tree containing all inner vertices and exactly one outer vertex. Each edge in each tree is oriented from a vertex to its parent and the single outer vertex of each tree has only incoming edges and therefore is the root of the tree. We will denote the root of each tree T_i by r_i , $0 \leq i \leq 2$. The three roots r_0 , r_1 , and r_2 are distinct and their counterclockwise order on the external boundary is 0, 1, and 2. In the following we denote a Schnyder realization $\{T_0, T_1, T_2\}$ of G as \mathcal{T} .

Let G be a maximal plane graph equipped with a Schnyder realization \mathcal{T} . Let v be an internal vertex and let $P_i(v)$, $0 \leq i \leq 2$ be the oriented path of T_i from v to r_i . The path $P_i(v)$ is called the *i-path starting at v*. For $i \neq j$ ($0 \leq i, j \leq 2$), the two paths $P_i(v)$ and $P_j(v)$ only share the vertex v . Thus, for each internal vertex v , the three paths $P_0(v)$, $P_1(v)$, and $P_2(v)$ divide G into three regions $R_0(v)$, $R_1(v)$, and $R_2(v)$, where $R_i(v)$ ($0 \leq i \leq 2$) denotes the vertices that are inside the cycle $P_{i-1}(v) \cup P_{i+1}(v) \cup (r_{i-1}, r_{i+1})$ (where indices are taken modulo 3). Notice that the regions $R_0(v)$, $R_1(v)$, and $R_2(v)$ do not contain the vertices of $P_0(v)$, $P_1(v)$, and $P_2(v)$ ¹. The following lemmas can be proven by exploiting the properties of Schnyder realization.

Lemma 3 *Let G be a maximal plane graph equipped with a Schnyder realization \mathcal{T} and let v be an internal vertex of G . Each $P_i(v)$ ($0 \leq i \leq 2$) is an induced path of G .*

Proof. Let $v = u_0, u_1, \dots, u_k = r_i$ be the vertices of $P_i(v)$ in the order they appear walking along $P_i(v)$ from v to r_i . Suppose, as a contradiction that $P_i(v)$ is not induced. This means that $P_i(v)$ has a chord, i.e., an edge connecting two non consecutive vertices of $P_i(v)$. We first observe that the chord must be completely contained either in the region $R_{i+1}(v)$ or in the region $R_{i-1}(v)$ (except for the endvertices), because otherwise either r_i would not be on the external boundary, or the chord would be crossed by $P_{i+1}(v)$ and by $P_{i-1}(v)$. Also, the chord cannot be an edge colored i , otherwise there would be a cycle in T_i . As a consequence the chord cannot be incident to vertex r_i (all the edges incident to r_i have color i except the two on the external boundary). For each vertex u_j ($0 < j < k$) the edge (u_j, u_{j+1}) is the unique outgoing edge of u_j colored i , while (u_{j-1}, u_j) is an incoming edge of u_j colored i . By the properties of the Schnyder realization the outgoing edge of u_j colored $i+1$ appears between (u_j, u_{j+1}) and (u_{j-1}, u_j) in the counterclockwise order around u_j , while the incoming edges colored $i+1$ appears between (u_{j-1}, u_j) and (u_j, u_{j+1})

¹In the literature these regions are usually defined as including also $P_0(v)$, $P_1(v)$, and $P_2(v)$. For our purposes, however, it results more useful to use this different definition.

in the counterclockwise order around u_j . In other words, the outgoing edge of u_j colored $i + 1$ and the incoming edges of u_j colored $i + 1$ are on different sides of the path $P_i(v)$. The same is true also for the edges colored $i - 1$. This implies that there cannot be a chord colored $i + 1$ or $i - 1$ connecting two vertices u_j and u_l with $0 < j, l < k$. The only possibility remaining is that the chord connects v to a vertex u_j with $0 < j < k$ and, as already said, it is colored either $i + 1$ or $i - 1$. The chord cannot be an outgoing edge of v otherwise one of the two paths $P_{i+1}(v)$ and $P_{i-1}(v)$ would share a vertex with $P_i(v)$, which is not possible. So the chord $e = (v, u_j)$ must be an incoming edge of v colored $i + 1$ or $i - 1$. Suppose it is colored $i + 1$ (the other case is symmetric). Then e must be contained in the region $R_{i+1}(v)$ (it must appear counterclockwise between the outgoing edge of v colored i and the outgoing edge of v colored $i - 1$). On the other hand, e is also an outgoing edge of u_j colored $i + 1$ and therefore it must be contained in the region $R_{i-1}(v)$ (it must appear counterclockwise between (u_j, u_{j+1}) and (u_{j-1}, u_j)). But, as already said, a chord must be contained in only one region, and therefore we have a contradiction. \square

Lemma 4 *Let G be a maximal plane graph equipped with a Schnyder realizer \mathcal{T} and let v be an internal vertex of G . The subgraph H_{ij} of G induced by $P_i(v)$, $P_j(v)$ ($0 \leq i, j \leq 2$) is an outerpath with the edge (r_i, r_j) on the external boundary of G . Also, $P_i(v)$ and $P_j(v)$ are both side paths of H_{ij} with respect to (r_i, r_j) .*

Proof. Let C be the cycle $P_i(v) \cup P_j(v) \cup (r_i, r_j)$. Since $P_i(v)$ and $P_j(v)$ are both induced paths, then every edge of H_{ij} that is not in C connects a vertex of $P_i(v)$ to a vertex of $P_j(v)$. If one of such edges was outside C , then it would be crossed by path $P_k(v)$ ($k \neq i, k \neq j$). Since all edges not in C are inside C , then H_{ij} is outerplanar. Since each edge not in C connects a vertex of $P_i(v)$ to a vertex of $P_j(v)$, then H_{ij} is an outerpath and both $P_i(v)$ and $P_j(v)$ are its side paths. Clearly, edge (r_i, r_j) is on the external boundary of G and both, $P_i(v)$ and $P_j(v)$ are side paths with respect to (r_i, r_j) . \square

A sequence of nested separating triangles of a maximal plane graph G is a set of k separating triangles t_1, t_2, \dots, t_k such that $t_i \succ t_{i+1}$, for $i = 1, 2, \dots, k - 1$.

Lemma 5 *Let G be an n -vertex maximal plane graph. If G contains a sequence of k nested separating triangles, then G contains an external outerpath with a side path of size at least $\lceil (k + 2)/3 \rceil$.*

Proof. Let $t_1 \succ t_2 \succ \dots \succ t_k$ be the sequence of nested separating triangles of G (see Figure 3(a)). Since t_k is a separating triangle, there exists a vertex u which lies inside t_k . Let \mathcal{T} be a Schnyder realizer of G . Since the three paths $P_0(u)$, $P_1(u)$, and $P_2(u)$ of G are disjoint, each path visits each triangle t_i ($i = 1, 2, \dots, k$) exactly once. Triangles t_1, \dots, t_k contain in total at least $k + 2$ distinct vertices. Therefore at least one among $P_0(u)$, $P_1(u)$, and $P_2(u)$ contains at least $\lceil (k + 2)/3 \rceil$ vertices. Suppose it is $P_0(u)$. By Lemma 4, the graph H_{01} induced by the vertices of $P_0(u)$ and $P_1(u)$ is an external outerpath of G with a side path of size at least $\lceil (k + 2)/3 \rceil$ (the same is true for the graph H_{02} induced by the vertices of $P_0(u)$ and $P_2(u)$). \square

3.2.2 Large Induced Outerpath in 4-connected Graphs

The proof that a 4-connected graph contains a large induced outerpath relies on the following result by Thomassen [26]. A planar drawing of G such that all edges (except four) are vertical or horizontal straight-line segments and such that the four exceptional edges each consist of a vertical and a horizontal straight-line segment is called *rectangular representation* of G .

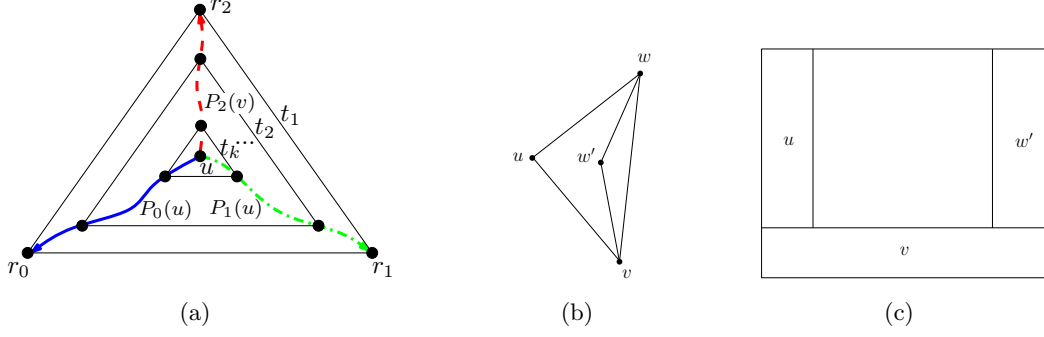


Figure 3: (a) Illustration for the proof of Lemma 5. (b)-(c) Illustration of the Theorem 4.

Theorem 4 (Thomassen [26]) *Let G be a 4-connected maximal plane graph and let uvw and wwv' be two triangles of G . Then G^* has a rectangular representation such that w corresponds to the unbounded face and such that u , v , and w' correspond to rectangles as shown in Figure 3(c).*

Lemma 6 *Let G be an n -vertex 4-connected maximal plane graph. Then G contains an external outerpath with a side path of size at least $\lceil \log_2(n-1)/4 \rceil$.*

Proof. Let u, w, v be the external boundary of G and let w' be the vertex of G , such that w, v, w' is a face of G . By Theorem 4, G^* has a rectangular representation such that w corresponds to the unbounded face and such that u , v , and w' correspond to rectangles as shown in Figures 3(c) and 3(b). I.e., G is the dual of a subdivision of a rectangle into $n-1$ rectangles. In [27] it is shown that there exists a horizontal or vertical line (called a *stabber*) that intersects at least $\lceil \log_2(n-1)/4 \rceil$ rectangles in this subdivision. The rectangles stabbed by a stabber represent an induced path P of G .

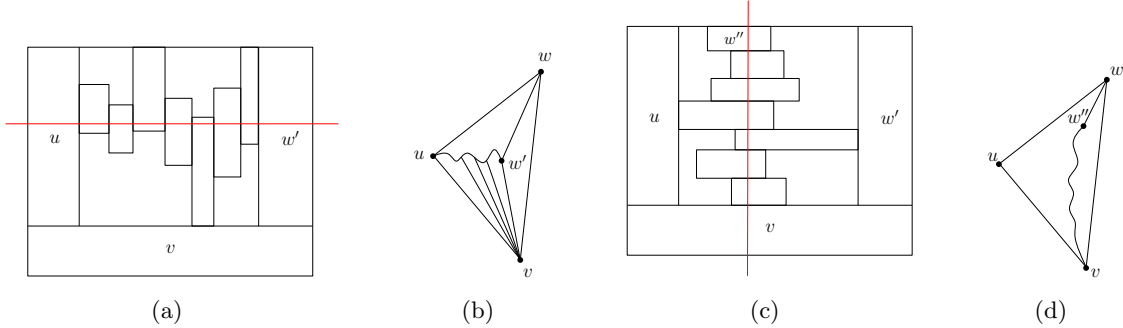


Figure 4: Illustration of the proof of Lemma 6. (a)-(b) Long stabber is horizontal. (c)-(d) Long stabber is vertical.

In the following we show that G contains an external outerpath that has P as a side path. We consider two cases based on whether the long stabber is vertical or horizontal.

Long stabber is horizontal Refer to Figures 4(a)- 4(b). Path P connects u to w' and edges (w', v) and (u, v) belong to G . Let C be the cycle $P \cup (w', v) \cup (u, v)$ and let H be the subgraph induced by the vertices in C . Since P is an induced path, each edge of H that does not belong to C is incident to v . From the rectangular representation, it is immediate to see that all the edges connecting v to P lie between (v, u) and (v, w') in the clockwise circular

order around v , i.e., they are inside the cycle C . Thus, H is an outerpath; it has an extremal edge (i.e., the edge (u, v)) on the external boundary of G ; and P is a side path of H with respect to (u, v) .

Long stabber is vertical Refer to Figures 4(c)- 4(d). Let w'' be the vertex representing the highest rectangle stabbed by the stabber. Edge (w'', w) belongs to G (recall that w is represented by the unbounded face). Moreover, vertex w is not adjacent to any other vertex of P except for v . Let C be the cycle $P \cup (w'', w) \cup (w, v)$. Since P is an induced path and there is no edge connecting w to a vertex of P distinct from v , the subgraph H induced by the vertices of C coincides with C itself, i.e., H is a cycle. Thus, H is an outerpath; it has an extremal edge (i.e., the edge (v, w)) on the external boundary of G ; and P is a side path of H with respect to (u, v) .

□

3.3 Putting All Together

We are now ready to combine the results of Section 3.1 and 3.2 in order to prove Theorem 1. To this aim, we use the following decomposition tree.

4-block tree. Let G be a maximal plane graph. A 4-block tree [28] T of G is a directed tree containing a vertex for each triangle of G (refer to Figure 5.a-b). Triangle of G that corresponds to a vertex μ of T is denoted by Δ_μ . T contains edge directed from its vertex μ to its vertex ν if $\Delta_\nu \prec \Delta_\mu$ and there is no triangle Δ in G such that $\Delta_\nu \prec \Delta \prec \Delta_\mu$. We assume that T is rooted at its source i.e. at the vertex which corresponds to the external face of G . Note that the leaves of T correspond to the internal faces of G . The subgraph of G induced by the vertices of Δ_μ and all the vertices of the children of μ is called the *pertinent* graph of μ and is denoted by G_μ (see Figure 5(c)).

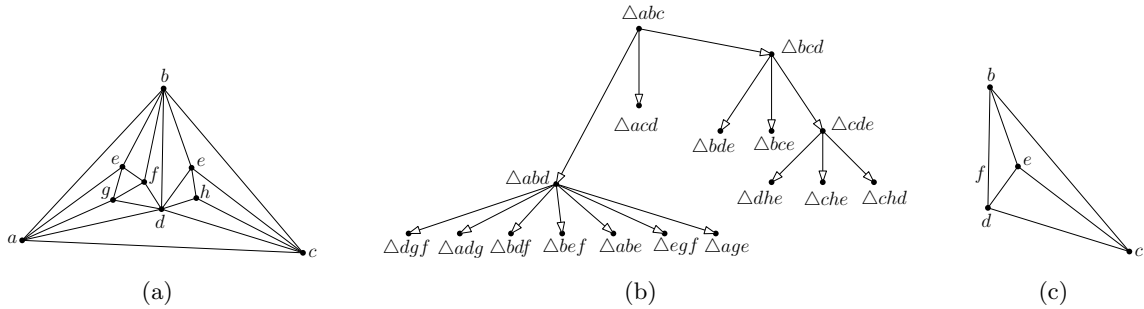


Figure 5: (a) A plane graph G . (b) An extended 4-block tree of G . (c) Pertinent graph G_μ of node $\mu = \Delta_{bcd}$

Theorem 5 *In every n -vertex maximal plane graph G there exists a maximal plane subgraph G' that contains an external outerpath H with a side path of size at least $\left\lceil \frac{\sqrt{\log_2 n - 1}}{2} \right\rceil$. Both G' and H can be computed in $O(n)$ -time.*

Proof. Let G be a n -vertex maximal plane graph and let T be its triangle decomposition tree. Let $\alpha(n)$ be a function of n to be specified later. We consider two cases based on whether the degree of T is bounded by $\alpha(n)$ or not.

The maximum outdegree of T is at most $\alpha(n)$. Since G is a maximal plane graph it contains exactly $2n - 4$ faces. Thus, T contains $2n - 4$ leaves and therefore, the height of T is at least $\left\lceil \log_{\alpha(n)}(2n - 4) \right\rceil$. This means that G contains a sequence of at least $k = \left\lceil \log_{\alpha(n)}(2n - 4) \right\rceil + 1$ nested separating triangles. By Lemma 5, G contains an external outerpath H with a side path of size at least $\left\lceil \frac{k+2}{3} \right\rceil = \left\lceil \frac{\left\lceil \log_{\alpha(n)}(2n-4) \right\rceil + 3}{3} \right\rceil$.

There exists a vertex of T with outdegree at least $\alpha(n) + 1$. Let μ be such a vertex. Then the pertinent graph G_μ of μ contains at least $\alpha(n) + 1$ faces. Since G_μ is maximal planar graph, it contains at least $n_\mu = \left\lceil \frac{\alpha(n)+1}{2} \right\rceil + 2$ vertices. By Lemma 6, G_μ contains an external outerpath H with a side path of size at least $\left\lceil \frac{\log_2(n_\mu-1)}{4} \right\rceil = \left\lceil \frac{\log_2(\lceil (\alpha(n)+1)/2 \rceil + 1)}{4} \right\rceil$.

Thus, the size of the side path of H is

$$h = \min \left\{ \left\lceil \frac{\left\lceil \log_{\alpha(n)}(2n - 4) \right\rceil + 3}{3} \right\rceil, \left\lceil \frac{\log_2(\lceil (\alpha(n) + 1)/2 \rceil + 1)}{4} \right\rceil \right\},$$

By setting $\alpha(n) = 2^{\sqrt{\lg n}}$, we get that $h \geq \min \left\{ \left\lceil \frac{\left\lceil \sqrt{\log_2 n} + 3 \right\rceil}{3} \right\rceil, \left\lceil \frac{\sqrt{\log_2 n - 1}}{4} \right\rceil \right\} = \left\lceil \frac{\sqrt{\log_2 n - 1}}{4} \right\rceil$.

We prove now that G' and H can be computed in $O(n)$ time. First, we compute the Schnyder realizer of G in $O(n)$ time [24]. Then, we compute the longest path of each of the trees T_0, T_1 , and T_2 and take the longest of the three. If such a path π has length at least $\left\lceil \frac{\sqrt{\log_2 n - 1}}{4} \right\rceil$ we are in the first case above. The computed path π is a path $P_i(u)$ for some internal vertex u of G . H is the subgraph induced by $P_i(u)$ and one of the other two paths $P_{i-1}(u)$ and $P_{i+1}(u)$ (in this case G' coincides with G). If π is not long enough, we are in the second case above. In this case we can find G_μ by computing the 4-block tree T , by performing a visit of T in order to find the node with the largest outdegree, and finally by computing the pertinent graph of μ . All this can be computed in $O(n)$ -time [28]. Once we have G_μ , we need to compute a rectangular representation of it with the properties described in the proof of Lemma 6 and then to compute the long stabber. A rectangular representation can be computed in $O(n)$ time [7] and the stabber can also be computed in $O(n)$ time [27]. In this case H is the subgraph induced by the path corresponding to the stabber together with a vertex of the external boundary that is adjacent to both endvertices of the path (in this case G' is G_μ). \square

Proof of Theorem 1. Let S be a set of distinct points in the plane such that $|S| \leq \left\lceil \frac{\sqrt{\log_2 n - 1}}{4} \right\rceil$ and let G be an n -vertex plane graph. If G is not maximal we can add dummy edges to it in order to make it maximal (these edges will be removed at end from the obtained drawing). Thus, without loss of generality, assume that G is maximal. By Theorem 5, in G there exists a maximal plane subgraph G' of G that contains an external outerpath with a side path of size at least $\left\lceil \frac{\sqrt{\log_2 n - 1}}{4} \right\rceil$. If G' coincides with G , then, by Lemma 1, G admits a geometric point-subset embedding on S . If G' does not coincide with G , then, by Lemma 1, G' admits a geometric point-subset embedding Γ' on S . Since G' is maximal plane, the external boundary of G' is a triangle t and it is drawn in the plane as a (geometric) triangle τ . All the other vertices and edges of G' are drawn inside τ . If we remove from G all the internal vertices of G' we obtain a new maximal plane graph G'' ; the

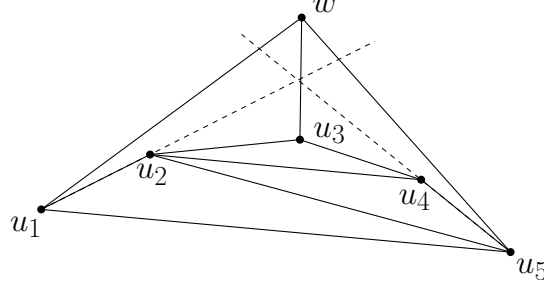


Figure 6: Illustration of the drawing technique of Lemma 7.

triangle t is a separating triangle of G and is an internal face of G'' . Let u be an internal vertex of G' and let \mathcal{T} be a Schnyder realizer of G . Since the three paths $P_0(u)$, $P_1(u)$, and $P_2(u)$ of G are disjoint and t is a separating triangle, each vertex of t belong to exactly one of the three paths $P_0(u)$, $P_1(u)$, and $P_2(u)$, which are induced paths by Lemma 3. It follows that G'' satisfies the conditions of Lemma 2 and therefore it admits a geometric point-subset embedding Γ'' on S that has τ as a subdrawing. The union of Γ' and Γ'' is a geometric point-subset embedding of G on S .

By Theorem 5, Lemma 1, and Lemma 2, the time complexity of the drawing algorithm is $O(k \log k + n)$ where k is the size of S . Since $k \leq \left\lceil \frac{\sqrt{\log_2 n - 1}}{4} \right\rceil$, we have that $O(k \log k) = O(n)$ which concludes the proof of theorem 1.

Let $F(n)$ be a function such that for every point set S of size $F(n)$, any planar graph admits a geometric point-subset embedding on S . Angelini et al. proved that $4 \leq F(n) \leq 2\lceil \frac{n}{3} \rceil + 2$, and ask whether $\lim_{n \rightarrow +\infty} F(n) = +\infty$. An immediate consequence of Theorem 1 is the following.

Corollary 1 $\lim_{n \rightarrow +\infty} F(n) = +\infty$.

Furthermore, Theorem 5 implies the following corollary which is, for maximal planar graphs, an improvement over the result by Aroha and Valencia [2] who showed that every 3-connected planar graph with n vertices and maximum vertex degree d contains an induced path of length $\Omega(\sqrt{\log_3 d})$.

Corollary 2 *Let G be an n -vertex maximal plane graph. Then G has an induced path with at least $\left\lceil \frac{\sqrt{\log_2 n - 1}}{4} \right\rceil$ vertices.*

4 One-sided Convex Point Sets

A *one-sided convex point set* S is a set of points in the plane that are in convex position and can be rotated so that the leftmost point and the rightmost point are adjacent in the convex hull of S . This restriction on the point set allows us to increase the size of the set itself. The structure of the proof of Theorem 2 is similar to that of Theorem 1.

4.1 Drawing tool

Lemma 7 *Let S be a one-sided convex point set with cardinality k . Let G be an n -vertex maximal plane graph that contains an induced biconnected outerplanar subgraph with at least k vertices and one edge on the external boundary of G . Then G admits a geometric point-subset embedding on S , which can be computed in $O(k \log k + n)$ time.*

Proof. If necessary, rotate S so that the leftmost and the rightmost point are adjacent in the convex hull of S . Let p_1, p_2, \dots, p_k be the points of S in the order they appear along the convex hull of S , with p_1 being the leftmost point and p_k being the rightmost point. Without loss of generality assume that each point p_i ($1 < i < k$) is above segment $\overline{p_1 p_k}$. Let H be the induced biconnected outerplanar subgraph of G and let C be external boundary of H . Without loss of generality assume that H has exactly k vertices (if not, we can augment S with additional points so to maintain S to be a one-sided convex position). So without loss of generality assume that H has exactly k vertices. Let P be the path obtained from C by removing the edge of H that is on the external boundary of G . Let u_1, u_2, \dots, u_k be the vertices of P in the order they appear along P . Map u_i to p_i and draw each edge (u_i, u_{i+1}) as the segment $\overline{p_i p_{i+1}}$ (see Figure 6 for an illustration of the drawing technique). Also add every other edge connecting two of the vertices of P . Since S is convex position, it is immediate to see that the resulting drawing has no crossings. Let w be the outer vertex of G that does not belong to H . Let ρ_1 be the straight line containing the segment $\overline{p_1 p_2}$ and let ρ_k be the straight line containing the segment $\overline{p_{k-1} p_k}$. Chose a point p that is above ρ_1 and above ρ_k , place w on p , and draw all edges connecting w to the vertices of H . It is easy to see that the obtained drawing Γ' has no crossing. Hence Γ' is a planar geometric embedding on S of the graph induced by the vertices in H plus w . Moreover, all the faces of the drawing are star-shaped polygons. In the planar embedding of G each vertex not yet drawn is inside one of the cycles represented by the faces of the computed drawing. Thus, the drawing can be completed by using Theorem 3 as described in the proof of Lemma 1. Similar to Lemma 1, the time spent to compute the drawing is $O(n)$, once the points are sorted. Thus, the time complexity is $O(k \log k + n)$. \square

4.2 Large Induced Biconnected Outerplanar Subgraph

We now show that any n -vertex maximal plane graph contains an induced biconnected outerplanar subgraph with at least $\lceil \sqrt[3]{n} \rceil$ vertices and one edge on the external boundary of G . To this aim we will use the Schnyder realizer to define three different partial orders on the vertices of G .

Let G be a maximal plane graph equipped with a Schnyder realizer \mathcal{T} . For each $i = 1, 2, 3$, we define a partial order on $V(G)$ by constructing a graph G_i by means of the following operations² (see Figure 7 for an illustration).

- We first remove from G all the internal edges with color different from i and $i + 1$;
- We then reverse the orientation of the edges with color i ;
- We color the outer edge (r_i, r_{i+1}) with color i and orient it from r_i to r_{i+1} ;
- We color the outer edge (r_i, r_{i-1}) with color i and orient it from r_i to r_{i-1} ;
- We color the outer edge (r_{i-1}, r_{i+1}) with color $i + 1$ and orient it from r_{i-1} to r_{i+1} .

The graph G_i is a directed acyclic graph with one source r_i and one sink r_{i+1} . Thus, it defines a partial order \prec_i for the vertices of G ; namely, $u \prec_i v$ if there is a directed path from u to v in G_i . Given a set X with a partial order \prec (also called a *partial ordered set*), a subset X' of X such that every two elements in X' are comparable with respect to \prec is called a *chain* of X . A subset X'' of X such that no two elements in X'' are comparable with respect to \prec is called an *antichain* of X . According to the definition of \prec_i , a chain χ of $V(G)$ with respect to \prec_i corresponds to a oriented path π in G_i . We say that π is the oriented path of G_i associated with χ . We prove now some properties of the partial orders \prec_i .

²Although defined in a different way, this graph is the same as the Frame graph defined in [6]

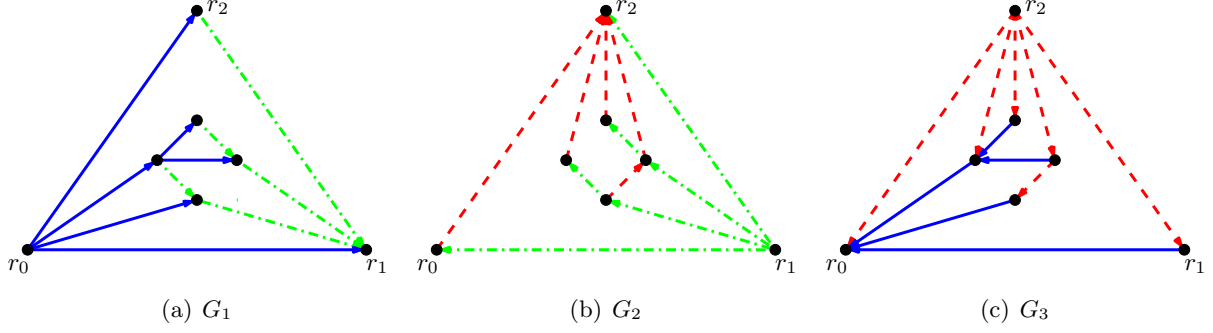


Figure 7: The three directed graphs used to define the three partial orders \prec_1 , \prec_2 , and \prec_3 .

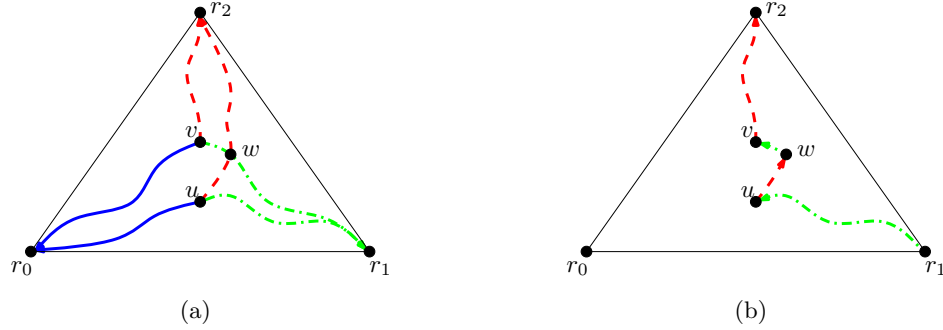


Figure 8: An illustration for the proof of Lemma 8. (a) The path $P_3(u)$ crosses the path $P_2(v)$. (b) The oriented path containing u and v in G_2 .

Lemma 8 *Let G be a maximal plane graph equipped with a Schnyder realizer \mathcal{T} . Let u and v be two internal vertices of G . Then u and v are comparable by at least one of the three partial orders \prec_1 , \prec_2 , and \prec_3 .*

Proof. If u belongs to a i -path starting at v for some i ($1 \leq i \leq 3$), then u and v are comparable with respect to both \prec_i and \prec_{i-1} (indices modulo 3). Namely, the concatenation of the reversed version of $P_i(v)$ with $P_{i+1}(v)$ is an oriented path of $G_i(v)$ containing both u and v , and therefore u and v are comparable with respect to \prec_i . Analogously, the concatenation of the reversed version of $P_{i-1}(v)$ with $P_i(v)$ is an oriented path of $G_{i-1}(v)$ containing both u and v , and therefore u and v are comparable with respect to \prec_{i-1} . Clearly, the same argument applies if v belongs to a i -path starting at u for some i ($1 \leq i \leq 3$).

Consider the case when u does not belong to any i -path starting at v and viceversa. It follows that u must be in one region $R_i(v)$ ($1 \leq i \leq 3$). Consider the i -path starting at u , $P_i(u)$. Since this path does not contain v , it must share a vertex w with either $P_{i-1}(v)$ or $P_{i+1}(v)$ (see Figure 8(a)). Suppose that w belongs to $P_{i-1}(v)$, the case when it belongs to $P_{i+1}(v)$ is analogous. Consider the path P consisting of the concatenation of: (i) the path $P_{i-1}(u)$; (ii) the subpath of $P_i(u)$ from u to w ; (iii) the subpath of $P_{i-1}(v)$ from w to v ; (iv) the path $P_i(v)$. All edges of P colored $i-1$ are oriented towards r_{i-1} , while those colored i are oriented towards r_i (see Figure 8(b)). Thus reversing the edges of P colored $i-1$, we obtain an oriented path P' that is an oriented path of G_{i-1} and that contains both u and v . Thus, u and v are comparable with respect to \prec_{i-1} . \square

Lemma 9 *Let G be a maximal plane graph equipped with a Schnyder realizer \mathcal{T} . Let u be an*

internal vertex of G and let v be another vertex of G such that $v \in R_{i-1}(u)$. Then u and v are not comparable with respect to \prec_i .

Proof. If $v \in R_{i-1}(u)$, the two paths $P_i(u)$ and $P_i(v)$ have a vertex in common w_1 , and the paths $P_{i+1}(u)$ and $P_{i+1}(v)$ have a vertex in common w_2 . Thus, in G_i , there are two oriented paths π_1 from w_1 to w_2 : one, call it π_1 , containing u and the other one, call it π_2 , containing v . We prove now that there is no oriented path from v to u in G_i (the proof that there is no oriented path from u to v is analogous). Suppose that in G_i there is an oriented path π from v to u . If π does not share a vertex with $P_i(u) \cup P_{i+1}(u)$ nor with $P_i(v) \cup P_{i+1}(v)$, then it is completely contained in the (unoriented) cycle C formed by π_1 and π_2 . Let e_i be the first edge of $P_i(u)$ (this edge is the unique outgoing edge of u colored i in \mathcal{T}) and let e_{i+1} be the first edge of $P_{i+1}(u)$ (this edge is the unique outgoing edge of u colored $i+1$ in \mathcal{T}). Since the edges of π must be colored i or $i+1$, there should be an edge colored i or $i+1$ that follows e_i and precedes e_{i+1} in the counterclockwise order of the edges around u . But all edges that follows e_i and precedes e_{i+1} in the counterclockwise order around u are colored $i-1$. Suppose now that π shares a vertex w with $P_i(u) \cup P_{i+1}(u)$ and/or $P_i(v) \cup P_{i+1}(v)$. Vertex w cannot belong to path $P_i(u)$ because, in this case, w would precede both u and v according to \prec_i and therefore there would be a directed cycle in G_i . Analogously, w cannot be in $P_{i+1}(v)$ because, in this case, w would follow both u and v according to \prec_i and therefore there would be a directed cycle in G_i . The only possibility is that w belongs to either the subpath of $P_i(u)$ from u to w_1 , or to the subpath of $P_{i+1}(v)$ from v to w_2 (endpoints excluded). Suppose that w belongs to $P_i(u)$. Let w' and w'' be the vertices that precede and follow w along $P_i(u)$, respectively. In \mathcal{T} , edge (w', w) is an incoming edge of w colored i , while edge (w, w'') is the unique outgoing edge of w colored i . As a consequence, path π cannot pass through w “exiting” from cycle C . In order to exit from C , in G_i there should be an incoming edge of w that follows (w', w) and precedes (w, w'') in the counterclockwise order around w . In \mathcal{T} this edge should be either a incoming edge of w colored $i+1$ or an outgoing edge of w colored i , but none of the two is possible according to the properties of the Schnyder realizer. With a symmetric argument one can show that also when w belongs to $P_{i+1}(v)$, path π cannot pass through w “exiting” from cycle C . Thus, in both cases path π can only enter cycle C . On the other hand, path π cannot be completely outside cycle C (in this case it would cross the common subpath of $P_i(u)$ and $P_i(v)$, or the common subpath of $P_{i+1}(u)$ and $P_{i+1}(v)$, but we have already seen that it is not possible). It follows that path π enters u from inside C , i.e., there is an edge colored i or $i+1$ that follows e_i and precedes e_{i+1} in the counterclockwise order of the edges around u , but we have already seen that this is not possible. \square

Lemma 10 *Let G be a maximal plane graph equipped with a Schnyder realizer \mathcal{T} . Let u be an internal vertex of G and let v be another vertex of G that is comparable with u by \prec_i . If $v \prec_i u$ then $v \in R_{i+1}(u) \cup P_i(u)$. If $u \prec_i v$ then $v \in R_i(u) \cup P_{i+1}(u)$.*

Proof. We consider the case when $v \prec_i u$ (the case when $u \prec_i v$ can be proven symmetrically). We show that v must be in $R_{i+1}(u) \cup P_i(u)$. By Lemma 9, v cannot be in $R_{i-1}(u)$ nor in $P_{i-1}(u)$ (in this case u would be in $R_{i-1}(v)$). If v is in $P_{i+1}(u)$ then u and v are comparable with respect to \prec_i but $u \prec_i v$, so v cannot be in $P_{i+1}(u)$. Suppose that v is in $R_i(u)$. If $P_i(v)$ has a vertex in common with $P_{i-1}(u)$, then we are again in the case when u is in $R_{i-1}(v)$ and thus u and v are not comparable by Lemma 9. It follows that $P_i(v)$ must have a vertex w in common with $P_i(u)$. Consider the path P consisting of the concatenation of: (i) the path $P_i(u)$; (ii) the subpath of $P_{i+1}(u)$ from u to w ; (iii) the subpath of $P_i(v)$ from w to v ; (iv) the path $P_{i+1}(v)$. All edges of P colored i are oriented towards r_i , while those colored $i+1$ are oriented towards r_{i+1} . Thus reversing the edges

of P colored i , we obtain an oriented path P' that is an oriented path of G_i and that contains both u and v . Thus, u and v are comparable with respect to \prec_i . Note however that in this case $u \prec_i v$. Thus v cannot be in $R_i(u)$. The only possibility remaining is that v is in $R_{i+1}(u) \cup P_i(u)$. \square

In the next lemma we show that any maximal chain of $V(G)$ with respect to any partial order \prec_i defines an induced biconnected outerplanar subgraph with one edge on the external boundary of G .

Lemma 11 *Let G be a maximal plane graph equipped with a Schnyder realizer \mathcal{T} . Let χ be a maximal chain of $V(G)$ with respect to the partial order \prec_i , $1 \leq i \leq 3$ and let π be the oriented path associated with χ . The graph induced by the vertices of π is an induced biconnected outerplanar subgraph with one edge on the external boundary of G .*

Proof. First of all, notice that since G_i has a single source r_i and single sink r_{i+1} , then every maximal chain of $V(G)$ with respect to \prec_i has r_i as the first element and r_{i+1} as the last element. As a consequence π starts at r_i and ends at r_{i+1} . Also, since χ contains at least one internal vertex, π consists of all internal edges. This implies that it does not coincide with the single edge (r_i, r_{i+1}) . Thus, the concatenation of (the underlying undirected path of) π and of the edge (r_i, r_{i+1}) is a simple cycle C of G . We prove that the subgraph H induced by the vertices in C is a biconnected outerplanar graph. If C is the boundary of the external face, then H is a 3-cycle and therefore it is a biconnected outerplanar graph. If C is not the boundary of the external face of G , in order to prove that is a biconnected outerplanar graph, it is sufficient to show that all edges of H that are not in C are inside C in the planar embedding of G . Consider an edge (u, v) of H not in C . Vertices u and v must be two non-consecutive vertices of π . Without loss of generality assume $u \prec_i v$, i.e., u is encountered first when walking along π from r_i to r_{i+1} . Since we are considering an edge not in C , it must be $u \neq r_i$ or $v \neq r_{i+1}$ (or both). Since u and v are non-consecutive, there must be a vertex w such that $u \prec_i w \prec_i v$. By Lemma 10, u is in $R_{i+1}(w) \cup P_i(w)$ and v is in $R_i(w) \cup P_{i+1}(w)$. As a consequence, if edge (u, v) is not inside C , then it crosses path $P_{i-1}(w)$, which is not possible. \square

The next lemma shows that the subgraph of Lemma 11 has at least $\lceil \sqrt[3]{n} \rceil$ vertices.

Lemma 12 *Let G be a maximal plane graph equipped with a Schnyder realizer \mathcal{T} . $V(G)$ has a chain of size at least $\lceil \sqrt[3]{n} \rceil$ with respect to one of the three partial orders \prec_1 , \prec_2 , and \prec_3 .*

Proof. Notice that, in order to prove that $V(G)$ has a chain of size at least $\lceil \sqrt[3]{n} \rceil$, it is sufficient to show that $V(G)$ has a chain of size at least $\sqrt[3]{n}$ because the number of vertices in a chain is necessarily an integer number. Consider first the partial order \prec_1 . If $V(G)$ has a chain of size at least $\sqrt[3]{n}$ with respect to \prec_1 , then the statement is true. If not, by Mirsky's theorem [20] (the dual of Dilworth's theorem), $V(G)$ has a partition into at most $\sqrt[3]{n}$ antichains with respect to \prec_1 . Hence, one of this antichains must have at least $\frac{n}{\sqrt[3]{n}} = n^{2/3}$ vertices. Let U be such a set of vertices. Since the vertices in U form an antichain of $V(G)$ with respect to \prec_1 , then, by Lemma 8, any two vertices in U must be comparable by \prec_2 or by \prec_3 . Consider \prec_2 restricted to the set U ; if U has a chain of size at least $\sqrt[3]{n}$ with respect to \prec_2 , then the statement is true. Otherwise, applying again Mirsky's theorem, U can be partitioned into at most $\sqrt[3]{n}$ antichains. One of this antichain must have at least $\frac{n^{2/3}}{n^{1/3}} = \sqrt[3]{n}$ vertices. Let U' be such a set of vertices. The vertices in U' are not comparable with respect to \prec_1 , neither with respect to \prec_2 . Thus by Lemma 8 they must be comparable by \prec_3 , i.e., they form a chain of U' with respect to \prec_3 . \square

Theorem 6 *Every n -vertex maximal plane graph G contains an induced biconnected outerplanar subgraph with at least $\lceil \sqrt[3]{n} \rceil$ vertices and one edge on the external boundary of G . Such a subgraph can be computed in $O(n)$ -time.*

Proof. Consider a Schnyder realizer \mathcal{T} of G and define the three partial orders \prec_1 , \prec_2 , and \prec_3 . By Lemma 12, $V(G)$ has a chain of size at least $\lceil \sqrt[3]{n} \rceil$ with respect to one of these partial orders. From Lemma 11 it follows that G contains an induced biconnected outerplanar subgraph H with at least $\lceil \sqrt[3]{n} \rceil$ vertices and one edge on the external boundary of G .

The Schnyder realizer \mathcal{T} can be computed in $O(n)$ time [24]. Using \mathcal{T} , the three graphs G_1 , G_2 , and G_3 can be also computed in $O(n)$ time. Computing the longest path in each of these graphs (which can be done in $O(n)$ time) we find the longest chain of $V(G)$ with respect to one of the three partial orders \prec_1 , \prec_2 , and \prec_3 . The subgraph induced by the vertices in this path is H . \square

Proof of Theorem 2 Let S be a one-sided convex point set of cardinality at most $\lceil \sqrt[3]{n} \rceil$ and let G be an n -vertex plane graph. If G is not maximal we can add dummy edges to it in order to make it maximal (these edges will be removed at end from the obtained drawing). Thus, without loss of generality, assume that G is maximal. By Theorem 6 G contains an induced biconnected outerplanar subgraph with at least $\lceil \sqrt[3]{n} \rceil$ vertices and one edge on the external boundary of G . By Lemma 7, G admits a geometric point-subset embedding on S .

By Theorem 6 and Lemma 7 the time complexity of the drawing algorithm is $O(k \log k + n)$ where k is the size of S . Since $k \leq \lceil \sqrt[3]{n} \rceil$, we have that $O(k \log k) = O(n)$ which concludes the proof of Theorem 2.

Theorem 2 can be used to solve a different problem, called the *allocation problem*. In the allocation problem [23] the input is an n vertex planar graph G and a point set X of cardinality n , and the goal is to compute a planar straight-line drawing of G such that as many vertices as possible are represented by points of X . The number of points fitting this purpose is denoted as $fit^X(G)$ and $fit(G) = \min_X \{fit^X(G)\}$, where the minimum is taken over all n -point sets X . Upper and lower bounds on $fit(G)$ are known for various classes of graphs [6, 14, 23]. We define $fit^C(G) = \min_X \{fit^X(G)\}$ where the minimum is taken over all X in one-sided convex position. Theorem 2 implies that $fit^C(G) \geq \lceil \sqrt[3]{n} \rceil$ for every planar graph G . This improves the best known lower bound on $fit(G)$ for general planar graphs which is $\sqrt[4]{\frac{n}{3}}$ [6].

Corollary 3 *For every n -vertex planar graph, $fit^C(G) \geq \lceil \sqrt[3]{n} \rceil$.*

5 Conclusions and Open Problems

In this paper we proved that a planar graph with n vertices admits a geometric point-subset embedding on every set of $\left\lceil \frac{\sqrt{\log_2 n - 1}}{4} \right\rceil$ points and on every one-sided convex point set of $\lceil \sqrt[3]{n} \rceil$ points on the plane. These are the first non-constant lower bounds for the size of arbitrary (one-sided convex) point sets on which planar graphs have geometric point-subsets embedding. In order to achieve this result we proved that every maximal planar graph contains as a subgraph an induced outerpath of size at least $\left\lceil \frac{\sqrt{\log_2 n - 1}}{4} \right\rceil$ and an induced biconnected outerplanar graph of size $\lceil \sqrt[3]{n} \rceil$. To the best of our knowledge, these are the first lower bounds for the size of a largest induced outerpath and biconnected outerplanar subgraph in planar graphs. It is known that not all planar graphs with n vertices admit a geometric point-subset embedding on a one-sided convex set

containing more than $2 \lceil \frac{n}{3} \rceil + 2$ points [1]. Thus closing the gap between lower bounds $\left\lceil \frac{\sqrt{\log_2 n} - 1}{4} \right\rceil$ and $\lceil \sqrt[3]{n} \rceil$, and upper bound $2 \lceil \frac{n}{3} \rceil + 2$ is an interesting open problem. From the graph-theoretical point of view it would be also interesting to study the upper bounds for the size of maximum induced biconnected outerplanar subgraph and induced outerpath in biconnected planar graphs. However, it is worth mentioning that an upper bound for the size of induced outerplanar subgraph does not provide us an upper bound for the size of convex point set that represent universal subset for planar graphs.

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